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SMOOTH ODD FIXED POINT ACTIONS ON $\mathbb{Z}_2$-HOMOLOGY SPHERES

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Abstract. Let G be S_5 or $SL(2,5)$ and leet Σ be a homology sphere with smooth G-action such that the G-fixed point set consists of odd-number points. Then the dimension of Σ could be restrictive. In this article, we report results on the dimension of Σ and on the tangential G-representation of a G-fixed point in Σ.

This is a report of a joint work with Shunsuke Tamura.

1. Review of known results

In the present article, G is a finite group and G-actions on manifolds should be understood as smooth G-actions. By various researchers, G-actions on spheres with finite G-fixed points have been studied.

Throughout the article, let A_n and S_n denote the alternating group and the symmetric group on n letters, respectively, and let C_n denote the cyclic group of order n. First we like to recall several results found so far.

(1) For $G = A_5$, there are G-actions on $\mathbb{Z}$-homology spheres Σ of dimension 3 such that $|\Sigma^G| = 1$, e.g. $\Sigma = S^3/SL(2,5)$.

(2) (E. Stein [30]) For $G = SL(2,5)$, there exist effective G-actions on the sphere S^7 (of dimension 7) such that $|S^G| = 1$.

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(3) (T. Petrie [26]) Let G be an abelian group of odd order possessing at least 3 non-cyclic Sylow subgroups. Then there exist effective G-actions on spheres S^n, for some integers n, such that $|S^G| = 1$.

(4) (E. Laitinen–M. Morimoto [11]) A finite group G has effective G-actions on spheres S^n, for some n, such that $|S^G| = 1$ if and only if G is an Oliver group, i.e. $n_G = 1$ cf. [24, 23].

(5) (A. Borowiecka [3]) Let $G = SL(2,5)$. Then S^8 does not admit any effective G-action satisfying $|S^G| = 1$.

(6) (M. Morimoto [15, 17, 18], A. Bak–M. Morimoto [1, 2]) Let $G = A_5$. Then there are effective G-actions on S^n satisfying $|S^G| = 1$ if and only if $n \geq 6$.

(7) (B. Oliver [23]) Let G be an Oliver group, and V_1, V_2, $\ldots$, V_m are $\mathcal{P}$-matched real G-modules such that $V_i^G = 0$ for $1 \leq i \leq m$. Then there are effective G-actions on spheres S and real G-modules W such that $S^G = \{x_1, \ldots, x_m, y_1, \ldots, y_m\}$ and $T_{x_i}(S) \cong V_i \oplus W \cong T_{y_i}(S)$ for $1 \leq i \leq m$.

(8) (M. Morimoto–K. Pawałowski [21], M. Morimoto [20]) Let G be a gap Oliver group, and V_1, V_2, $\ldots$, V_m are $\mathcal{P}$-matched real G-modules such that $V_i^N = 0$ for $1 \leq i \leq m$ and $N \trianglelefteq G$ with prime power index $|G : N|$. Then there are effective G-actions on spheres S and real G-modules W such that $S^G = \{x_1, \ldots, x_m\}$ and $T_{x_i}(S) \cong V_i \oplus W$ for $1 \leq i \leq m$.

2. Report of results

We define the sets T_G of integers, for several finite groups G, as follows.

- $T_{A_5} = [0..2] \cup \{4, 5\}$.
- $T_{SL(2,5)} = [0..6] \cup \{8, 9\}$.
- $T_{S_5} = [0..5] \cup [7..9] \cup \{13\}$.
- $T_{A_6} = [0..7] \cup [9..12] \cup \{14, 15\} \cup \{19, 20\}$.
- $T_{SL(2,9)} = [0..15] \cup [17..20] \cup \{22, 23\} \cup \{27\}$.
- $T_{S_6} = [0..15] \cup [17..20] \cup [22..24] \cup [27..29] \cup \{33\} \cup \{38\}$.

We would like to report the following results.

Theorem 2.1 (cf. [22])**.** *Let G be A_5 or $SL(2,5)$ (resp. S_5) and let Σ be a $\mathbb{Z}_2$-homology (resp. $\mathbb{Z}$-homology) sphere of dimension n in T_G. Then Σ never admits effective G-actions satisfying $|\Sigma^G| \equiv 1 \mod 2$.*

Related to Theorem 2.1, we remark the following:

(1) S. Tamura announced an interesting result: Let G be A_6 or $SL(2,9)$ (resp. S_6) and let Σ be a $\mathbb{Z}_2$-homology (resp. $\mathbb{Z}$-homology) sphere of dimension n in T_G. Then Σ never admits effective G-actions satisfying $|\Sigma^G| \equiv 1 \mod 2$.

(2) In a recent work of A. Borowiecka–P. Mizerka, they gave certain subsets I_G (possibly the empty set) of $[6..10]$ for finite groups G such that $|G| \leq 216$ or $G \cong A_5 \times C_k$ with $k = 3$, 5, or 7, and they claimed that if $n \in I_G$ then there is no G-action on S^n satisfying $|S^G| = 1$.

Theorem 2.2. *Let G be S_5 and n a non-negative integer. If n does not belong to T_G then there exist effective G-actions on S^n satisfying $|S^G| = 1$.*

For a G-manifold X and $m \in \mathbb{N}$, let X_0^G denote the set consisting of all G-fixed points x in X such that $\dim T_x(X)^G = 0$, and let $X^G(m)$ denote the set consisting of all G-fixed points x in X such that $T_x(X)$ contains an irreducible real G-submodule of dimension m, where $T_x(X)$ stands for the tangential G-representation at x ($\in X^G$) in X.

Theorem 2.3. *Let $G = A_5$ and Σ a $\mathbb{Z}_2$-homology sphere with G-action.*

(1) *If $|\Sigma_0^G| \equiv 1 \mod 2$ then $\Sigma^G(3) \neq \emptyset$.*
(2) *If $|\Sigma^G| < \emptyset$ then $|\Sigma^G(3)| \equiv |\Sigma^G| \mod 2$.*

Theorem 2.4. *Let $G = S_5$ and Σ a $\mathbb{Z}$-homology sphere with G-action. If $|\Sigma^G| < \infty$ then $\Sigma^G(6) = \Sigma^G$.*

Theorem 2.1 follows from Theorems 2.3 and 2.4.

3. Irreducible real G-representations and fixed-point-set dimensions

In this section, we give basic data to prove Theorems 2.2 and 2.3. For a real G-representation V we call data of pairs $(H, \dim V^H)$ *fixed-point-set dimensions*, where H ranges over a set of subgroups of G.

Case 1. Let $G = A_4$. The irreducible real G-representations (up to isomorphisms) are $\mathbb{R}$, $U_{3,1}$, $U_{3,2}$, U_4, and U_5, where $\dim U_{3,i} = 3$, and $\dim U_k = k$. The G-actions on $U_{3,1}$, $U_{3,2}$, U_4, and U_5 are effective. We tabulate fixed-point-set dimensions of irreducible real A_5-representations.

	E	C_2	C_3	C_5	D_4	D_6	D_{10}	A_4	A_5
$\mathbb{R}$	1	1	1	1	1	1	1	1	1
$U_{3,i}$ $(i = 1,\ 2)$	3	1	1	1	0	0	0	0	0
U_4	4	2	2	0	1	1	0	1	0
U_5	5	3	1	1	2	1	1	0	0

Case 2. Let $G = SL(2,5)$. The irreducible real G-representations (up to isomorphisms) are $\mathbb{R}$, $U_{3,1}$, $U_{3,2}$, U_4, U_5, $W_{4,1}$, $W_{4,2}$, W_8, and W_{12}, where $U_*^Z = U_*$, ${W_*}^Z = 0$, $\dim \mathbb{R} = 1$, $\dim U_{k,i} = k$, $\dim U_k = k$, $\dim W_{k,i} = k$, and $\dim W_k = k$. The G-actions on $W_{4,1}$, $W_{4,2}$, W_8, and W_{12} are effective.

Case 3. Let $G = S_5$. The irreducible real G-representations (up to isomorphisms) are $\mathbb{R}$, $\mathbb{R}_\pm$, $V_{4,1}$, $V_{4,2}$, $V_{5,1}$, $V_{5,2}$, and V_6, where $\dim \mathbb{R} = 1$, $\dim \mathbb{R}_\pm = 1$, $\dim V_{k,i} = k$, and $\dim V_6 = 6$. The G-actions on $V_{4,1}$, $V_{4,2}$, $V_{5,1}$, $V_{5,2}$, and V_6 are effective. The characters of them are as follows.

	e	$(4,5)$	$(1,2)(4,5)$	$(1,2,3)$	$(1,2,3,4)$	$(1,2,3,4,5)$	$(1,2,3)(4,5)$
$\mathbb{R}$	1	1	1	1	1	1	1
V_1	1	-1	1	1	-1	1	-1
V_4	4	-2	0	1	0	-1	1
W_4	4	2	0	1	0	-1	-1
V_5	5	-1	1	-1	1	0	-1
W_5	5	1	1	-1	-1	0	1
V_6	6	0	-2	0	0	1	0

We tabulate fixed-point-set dimensions of irreducible real S_5-representations.

	S_5	A_5	S_4	$\mathfrak{F}_{20}$	$S_3\mathfrak{C}_2$	A_4	D_{10}	$\mathfrak{D}_8$	S_3
$\mathbb{R}$	1	1	1	1	1	1	1	1	1
$\mathbb{R}_\pm$	0	1	0	0	0	1	1	0	0
$V_{4,1}$	0	0	0	0	0	1	0	0	0
$V_{4,2}$	0	0	1	0	1	1	0	1	2
$V_{5,1}$	0	0	0	1	0	0	1	1	0
$V_{5,2}$	0	0	0	0	1	0	1	1	1
V_6	0	0	0	0	0	0	0	0	1

	D_6	$\mathfrak{C}_6$	C_5	$\mathfrak{D}_4$	D_4	$\mathfrak{C}_4$	C_3	$\mathfrak{C}_2$	C_2	E
$\mathbb{R}$	1	1	1	1	1	1	1	1	1	1
$\mathbb{R}_\pm$	1	0	1	0	1	0	1	0	1	1
$V_{4,1}$	1	1	0	0	1	1	2	1	2	4
$V_{4,2}$	1	1	0	2	1	1	2	3	2	4
$V_{5,1}$	1	0	1	1	2	2	1	2	3	5
$V_{5,2}$	1	1	1	2	2	1	1	3	3	5
V_6	0	1	2	1	0	1	2	3	2	6

Here D_m are dihedral subgroups of order m contained in A_5, C_m are cyclic subgroups of order m contained in A_5, $\mathfrak{F}_{20}$ is a subgroup of order 20 not contained in A_5, $\mathfrak{D}_m$ are dihedral subgroups of order m not contained in A_5, and $\mathfrak{C}_m$ are cyclic subgroups of order m not contained in A_5.

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